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13. ABSTRACT

A mainly expository survey prepared for an invited talk at the International Colloquium on Combinatorial Theories, Rome 1973. Several rather old problems in combinatorial geometry have recently been solved, mostly within the frameworks of either the theory of convex polytopes, or that of arrangements of lines or curves. Among those surveyed: Solution of Brunel's (1897) problem about the number of hexagons in 3-valent planar maps with precisely 3 digons; substantial results on Sylvester's (1867) problem about the number of collinear triplets possible with n points; solution and extension of de Rocquigny's (1897) problem about the number of points of tangency in systems of mutually non-overlapping circles; corrections and generalizations of previously published assertions about Venn diagrams (1880). Among the new results established are: sharpenings of Wernicke's(1904) and Kotzig's(1955) results about edges with endpoints of low valence in 3-dimensional convex polyhedra, and substantial improvements of previously known results on a problem of Erdös (1962) on collinear multiplets of points.

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Branko Grünbaum:

NEW VIEWS ON SOME OLD QUESTIONS OF COMBINATORIAL GEOMETRY *)

1. Introduction. The main motivation for this talk is my feeling that geometric aspects of combinatorial problems (and combinatorial aspects of geometric problems) are receiving less attention than they deserve - be the merit determined by intrinsic interest or by influence on and inspiration for other parts of combinatorics. The thinking of some of the most prominent founders of combinatorics (like L. Euler, J. Steiner, and T. P. Kirkman) was to a large extent geometric, and my thesis is that much that is valuable can be found by following their lead.

Although geometry and combinatorics interact at many levels and in a multitude of ways, I shall examine here only two such areas. This is not caused by any objective assessment of their value, but by the accident of my having recently been interested in them, or having heard of relevant new results. As will be apparent, many new problems - geometric as well as purely combinatorial - are suggesting themselves very naturally.

The two areas we shall discuss are convex polytopes, and arrangements. They are both concerned with phenomena in the (real) Euclidean plane or n-space, but their ramifications are very farflung. In each area, the coherent body of available geometric facts offers the possibility of obtaining interrelated points of view on a whole family of problems, and naturally leads to unexplored avenues in many directions. It may also suggest methods of proof, or instructive examples. I hope that the cases discussed in the following pages will provide ample illustrations, and possibly induce other workers to try their hands at solving the remaining open questions.

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2. Convex polytopes. The combinatorial aspects of the theory of polytopes appears to have been totally alien to the mathematicians of antiquity and the middle ages. Euler seems to have been the first to show genuine interest for it and to obtain non-trivial results (Euler characteristic, etc.). A large part of the interactions of the theory of 3-polytopes (3-dimensional convex polytopes) and combinatorics (especially graph theory) is based on the following theorem of Steinitz [1922] (see Grünbaum [1970; Section 1.2] for details and references to the literature):

A graph G is isomorphic to the graph defined by the vertices and edges of some 3-polytope if and only if G is planar and 3-connected.

To discuss the first group of graph-theoretic results and problems related to convex polytopes we recall the following theorem of V. Eberhard [1891]:

Given non-negative integers $p_3, p_4, p_5, p_7, p_8, \ldots, p_n$ with the property

$$(*) \qquad \sum_{3 \le k \ne 6}^{n} (6-k)\, p_k = 12 \quad ,$$

there exists a 3-valent 3-polytope P realizing $p_3, \ldots, p_n$, that is, such that $p_k(P) = p_k$ for all $k \ne 6$, where $p_k(P)$ is the number of k-gonal faces of P.

Following a simpler proof of Eberhard's theorem given in Grünbaum [1967; Section 13.3], a large number of analogues and extensions of Eberhard's theorem to planar graphs (and to graphs imbedded in other 2-manifolds) was obtained (for details and references see Grünbaum [1970; Section 1.3]). It should be noted that (using Steinitz's theorem, and duality for planar graphs) Eberhard's theorem may be interpreted as yielding information on the valence sequences ("degree sequences") of triangulations of the plane. It is well known that only the most rudimentary results have been found so far about valence sequences of arbitrary planar graphs, although that problem has been proposed (in the dual formulation) already by Sainte-Marie [1895].

There has been appreciable interest in the set of values of $p_6(P)$ possible (in Eberhard's theorem or its graph-theoretic analogues) for a given sequence $p_3,\ldots,p_n$. The first non-trivial result, establishing a special case of a conjecture of Eberhard [1891], was found in Grünbaum-Motzkin [1963]:

_If_ $p_3 = 4$ _and_ $p_j = 0$ _for_ $j \neq 3,6$, _then the set of values of_ $p_6(P)$ _possible in realizations of the sequence consists precisely of all the even integers different from_ 2. (For far-reaching generalizations of this result see Gallai [1971] and the references given there.)

The behaviour of the set $\{p_6\}$ in this case was established as typical by the following recent result of Fisher [1973]:

If a sequence $p_3,\ldots,p_n$ _satisfies_ (*), _the set of values of_ p_6 _possible in realizations of the sequence contains either all sufficiently large even integers, or all sufficiently large odd ones._

If $p_k(P)$ is interpreted as the number of k-gonal countries of the (2-connected) 3-valent planar graph P, then $p_2(P)$ is also meaningful, and Eberhard's theorem and the condition (*) may be extended to sequences that start with p_2.

A conjecture made already by G. Brunel [1897] (and independently also by Malkevitch [1970]) deals with such a case; it was recently established by Grünbaum-Zaks [1973] where it was proved together with a number of analogous results:

_If_ $p_2 = 3$ _and_ $p_j = 0$ _for_ $j \neq 2, 6$ _the set of values of_ p_6 _possible in realizations of this sequence is precisely the set of integers representable in the form_ $p_6 = u^2 + uv + v^2 - 1$, _where_ u, v _are non-negative integers and_ $(u,v) \neq (0,0)$.

This result shows that for $p_2 \neq 0$ the behaviour of $\{p_6\}$ is drastically different than in the case covered by Fisher's theorem.

Among the open problems related to these results are:

What are the analogues of Fisher's theorem if $p_2 \neq 0$; what if even p_1 is different from zero ?

Are there analogues of Eberhard's theorem for graphs imbedded in 2-manifolds if digons (and monogons) are permitted ? If digons (and monogons) are allowed, planar graphs may have all vertices of arbitrarily high valence. Zaks [1973] has established an Eberhard-type theorem for 6-valent planar graphs (in which case the number p_3 of triangles is not determined), but many natural extensions are still unexplored.

* * *

The second group of graph-theoretic problems derived from results on polytopes starts from the relatively recent (and unfortunately rather little known) beautiful theorem of Kotzig [1955] (for a more readily accessible proof see Grünbaum [1973a]):

Every 3-polytope P has an edge such that the sum of the valences of its endpoints is at most 13; the number 13 is best possible.

Kotzig's result, which can obviously be translated to deal with 3-connected planar graphs, has recently been strengthened in several directions. Jucovič [1973] has shown that every P has at least 3 edges of the type discussed, and at least 6 if P is simplicial (that is, if all the faces of P are triangles).

In order to describe some other results and open problems, let us denote by $e_{j,k} = e_{j,k}(P)$ the number of those edges of the 3-polytope or planar graph P which have one endpoint of valence j , the other of valence k . (The fact that $e_{5,5} + e_{5,6} > 0$ for planar triangulations of minimal valence 5 was established by Wernicke [1904] in connection with reductions of the four-color problem. Strengthenings of this result - which is less deep than Kotzig's - have also been given by Jucovič [1973]).)

A new improvement of Kotzig's theorem is:

If P is a triangulation of the plane with ≥ 4 vertices (or, equivalently, a simplicial 3-polytope) such that $e_{j,k} = 0$ whenever $j + k \leq 12$, then $e_{3,10} \geq 60$.

To prove this result we recall that the proof of Kotzig's theorem in Grünbaum [1973a] actually established that under the assumptions of the theorem

$$(**) \qquad e_{3,10} \geq 24 + 3v_{10} ,$$

where v_{10} is the number of 10-valent vertices. But this inequality has a rather interesting self-improving character. Since $e_{3,10} \geq 24$, and a 10-valent vertex is adjacent to at most five 3-valent ones, it follows that $v_{10} \geq 5$, so that $e_{3,10} \geq 39$ by (**). Then $v_{10} \geq 8$, so $e_{3,10} \geq 48$, thus $v_{10} \geq 10$ and then $e_{3,10} \geq 54$. Therefore $v_{10} \geq 11$, $e_{3,10} \geq 57$, and finally $v_{10} \geq 12$ and $e_{3,10} \geq 60$.

The inequality $e_{3,10} \geq 60$ is somewhat remarkable since it is best possible in the following strong sense: Equality holds for the 3-polytope obtained by placing 20 small pyramids on the faces of the icosahedron, as well as for infinitely many other 3-polytopes.

The last result and the theorem of Jucović [1973] lead to the idea that a relation of the type $\sum_{j+k \leq 13} \alpha_{j,k} e_{j,k} \geq 1$ should hold for every simplicial 3-polytope. More precisely, we conjecture that $6e^*_{3,10} + 2e_{4,9} + 10e_{3,9} + 8e_{4,8} + 15e^*_{3,8} + e_{5,7} + 12e_{4,7} + 20e_{3,7} + 36e_{6,6} + 6e^*_{5,6} + 15e^*_{4,6} + 30e^*_{3,6} + 12e^*_{5,5} + 21e^*_{4,5} + 36e^*_{3,5} + 36e_{4,4} + 45e^*_{3,4} + 60e^*_{3,3} \geq 360$. Not all the coefficients in this inequality are equally believable, but I feel reasonably sure about those indicated by an asterisk.

P. Erdős (private communication) has conjectured that Kotzig's theorem is valid for all planar graphs without loops or multiple edges and of minimal valence 3 .

Other open problems related to Kotzig's theorem include generalizations of the above to triangulations of (or to all 3-connected graphs imbeddable or 2-cell imbeddable in) 2-manifolds of higher genera. Kotzig's number for triangulations of the torus is easily seen to be 15, but nothing beyond that seems to be known. Even greater is the challenge to find meaningful analogues for suitable classes of higher-dimensional complexes.

* * *

Several other types of problems in graph-theory motivated by the theory of polytopes are discussed in Grünbaum [1973a] and [1973b]. Instead of repeating them here, we turn to the area of arrangements.

3. Arrangements. A general topic of investigation that has been frequently emerging in many variants for at least 150 years can be described as follows: What can we say about the various phenomena that may be observed in finite sets of points, lines, curves of various families, flats, or hyperplanes, in the real plane (Euclidean or projective) the sphere, or in the higher-dimensional spaces. The first - and rather limited - attempt to systematically survey the results known on this topic that we call "arrangements" (of points, lines, etc.) was made in the recent booklet Grünbaum [1972].

Historically, some of the problems on arrangements of points and lines generated interest in (or were associated with) various purely combinatorial structures - such as the Kirkman-Steiner triples, block designs, finite geometries, etc. Others led to investigations that are usually associated with algebra, or algebraic geometry, or with topology. It often turned out that in the modified or generalized setting the problems have a very elegant solution - but this does not (or at least it should not) obviate the need of trying to find answers to the original questions. For background, references, and material related to the topics discussed below the reader is generally refered to Grünbaum [1972], and for higher-dimensional material to Grünbaum [1971].

For a set A_n of n points in the plane, not all on one line, we shall denote by $m_j(A_n)$ the number of lines containing precisely j of the points. Several variants of the following problems were repeatedly mentioned and investigated by J. J. Sylvester between 1867 and 1893; many others were raised by other authors (see Grünbaum [1972; Chapter 2] and Burr-Grünbaum-Sloane [1973] for detailed references):

(i) Determine $m_2(n)$, the minimum of $m_2(A_n)$ when A_n varies over all (non-collinear) n-pointed sets in the plane.

(ii) For each $j \geq 3$ determine $m_j(n)$, the maximum of $m_j(A_n)$ when A_n varies over all n-pointed sets in the plane.

Concerning the first problem it is known that

$$3n/7 \leq m_2(n) \leq \alpha(n) ,$$

where

$$\alpha(n) = \begin{cases} 3,\ 3,\ 4,\ 6 & \text{for } n = 3,\ 4,\ 5,\ 13 \\ n/2 & n \text{ even},\ n \neq 4 \\ 3[n/4] & n \text{ odd},\ n \neq 3,\ 5,\ 13 . \end{cases}$$

It may be conjectured that $m_2(n) = \alpha(n)$ for all $n \geq 3$; this relation is known to hold for $3 \leq n \leq 14$ and $n = 16, 18, 22$. The bound $3n/7$ is due to Kelly-Moser [1958].

Concerning the case $j = 3$ of the second problem, some assertions of Sylvester's have recently been verified and strengthened in Burr-Grünbaum-Sloane [1973] where it was established that

$$\beta(n) \leq m_3(n) \leq \gamma(n) ,$$

with

$$\beta(n) = \begin{cases} 6,\ 16,\ 37,\ 52 & \text{for } n = 7,\ 11,\ 16,\ 19 \\ 1 + [n(n-3)/6] & \text{all other } n , \end{cases}$$

and

$$\gamma(n) = \left(\binom{n}{2} - m_2(n)\right)/3 .$$

It may be conjectured that $m_3(n) = \beta(n)$ for all n; this is known to be true for $4 \leq n \leq 12$ and for $n = 16$.

The determination of $m_3(n)$ is of interest, among others, because of its connections on the one hand to the Kirkman [1847] problem of maximizing the number of triplets on n elements, with no pair occuring twice, and on the other hand to the elementary algebraic geometry of cubic curves and to the Weierstrass elliptic functions.

The degree of our knowledge changes drastically when we turn to consider $m_j(n)$ for $j \geq 4$. The best result known is due to H. Croft and P. Erdős (unpublished):

<u>For each</u> $j \geq 3$ <u>there exists a constant</u> $c_j > 0$ <u>such that</u> $m_j(n) \geq c_j n^2$ <u>for all</u> $n > j$.

The proof of Croft and Erdős is based on the observation that for k sufficiently large, $c_j^* n^2$ lines intersect precisely j of the $n = k^2$ lattice points (x,y) with $0 \leq x,y < k$.

A related proof of the projectively dual statement (that n lines may be chosen so as to determine at least $c_j n^2$ vertices of multiplicity j - that is, through each of which pass exactly j of the lines) was indicated in Grünbaum [1972; p. 20]. We repeat it here briefly for $j = 4$, in order to show how to obtain an estimate of the form

(***) $$m_4(n) \geq n^2/64.$$

For $n \geq 6$, we consider the horizontal and the vertical lines, and those of slopes $+1$ or -1 , that pass through the points (x,y) of the integer lattice, with $0 \leq x < [n/6]$, $0 \leq y < [(n+3)/6]$. Then there are at most $3([n/6] + [(n+3)/6]) \leq n$ lines involved, and they determine $[n/6][(n+3)/6]$ quadruple points (x,y) ; the estimate (***) follows easily. It may readily be improved to $m_4(n) \geq n^2/40$.

Although it is probably not hard to show that

$$\bar{c}_j = \lim_{n \to \infty} m_j(n)/n^2$$

exists for each $j \geq 3$, no reasonable conjecture for the values $\bar{c}_j$ has been made. Even for $j = 4$, the estimates available are only

$$1/40 \leq \bar{c}_4 \quad 1/12 .$$

The upper bound follows from the following combinatorial result:

Let $m_4^a(n)$ denote the maximal number of quadruplets formed by n

elements, such that no pair occurs in more than one quadruplet. This combinatorial problem appears to be only partially solved (see Hanani [1961], Schönheim [1966], Hall [1967, p. 248], Niven [1970], Wilson [1970]) by the assertion that $m_4^a(n) \leq [[(n-1)/3]n/4] - \varepsilon(n)$, where $\varepsilon(n) = 1$ if $n \equiv 7$ or $n \equiv 10 \pmod{12}$ and $\varepsilon(n) = 0$ otherwise; equality holds for $n \equiv 0$ or $3 \pmod{12}$ and in some other cases, as well as for all sufficiently large n.

For small n, a certain amount of additional information on $m_4(n)$ is available; it has been collected in Table 1. (The author is greatly indebted to Mr. R. H. Macmillan for information on his unpublished results, and for permission to include here his figures, especially Figures 2 and 3.)

A particularly interesting variant of the problem on $m_j(n)$ is the following:

For each $j \geq 3$, determine $e_j(n)$, the maximum of $m_j(A_n)$ when A_n varies over all those n-pointed sets in the plane that contain no collinear $(j+1)$-tuple.

This problem was first raised by Erdős [1962], and Kárteszi [1963a] proved

(#) $\quad e_j(n) \geq d_j n \log n$ for some $d_j > 0$

by showing that $e_j(jn) \geq j\, e_j(n) + n$. He established this inequality by taking j copies of an A_n with $m_j(A_n) = e_j(n)$, and translating them by multiples of a suitable vector not parallel to any of the lines determined by A_n. The estimate (#) follows easily. Kárteszi [1963b] conjectured thet $e_4(n) \geq d_4 m^{3/2}$; we shall establish:

<u>For each</u> $j \geq 3$ <u>there exists</u> $d_j > 0$ <u>such that</u>

$$e_j(n) \geq d_j n^{(j-1)/(j-2)} .$$

The proof (of the projectively dual assertion) is by induction on j. For $j = 3$ it is enough to note that the systems of n lines and $\beta(n)$ triple points constructed in Burr-Grünbaum-Sloane [1973]

n	Maximal known $m_4(A_n)$		Maximal known $\tilde{m}_4(A_n)$
5	1	a	
6	1	a	
7	2	a	
8	2	a	
9	3	a	
10	5	a	
11	6	a	
12	7	a	
13	9		
14	10		
15	12	c	
16	15	See Figure 1a	
17			
18	18	See Figure 1b	
19			
20	20	b	
21	23	c	24
22	28	See Figure 2	
23			
24	29	c	33
25	30	b	
26	32	c	
27			
28	36	d	
29			
30	40	b	
31			
32			
33			
34	46	c	55
35	50	c	
36	55	See Figure 3	61

Table 1. Known lower bounds for $m_4(n)$.

(a) The number equals $m_4(n)$.

(b) See Figure 2.13 in Grünbaum [1972].

(c) Obtained by deleting or adding points to the arrangements in Figures 1, 2, or 3.

(d) Described by Ball [1960].

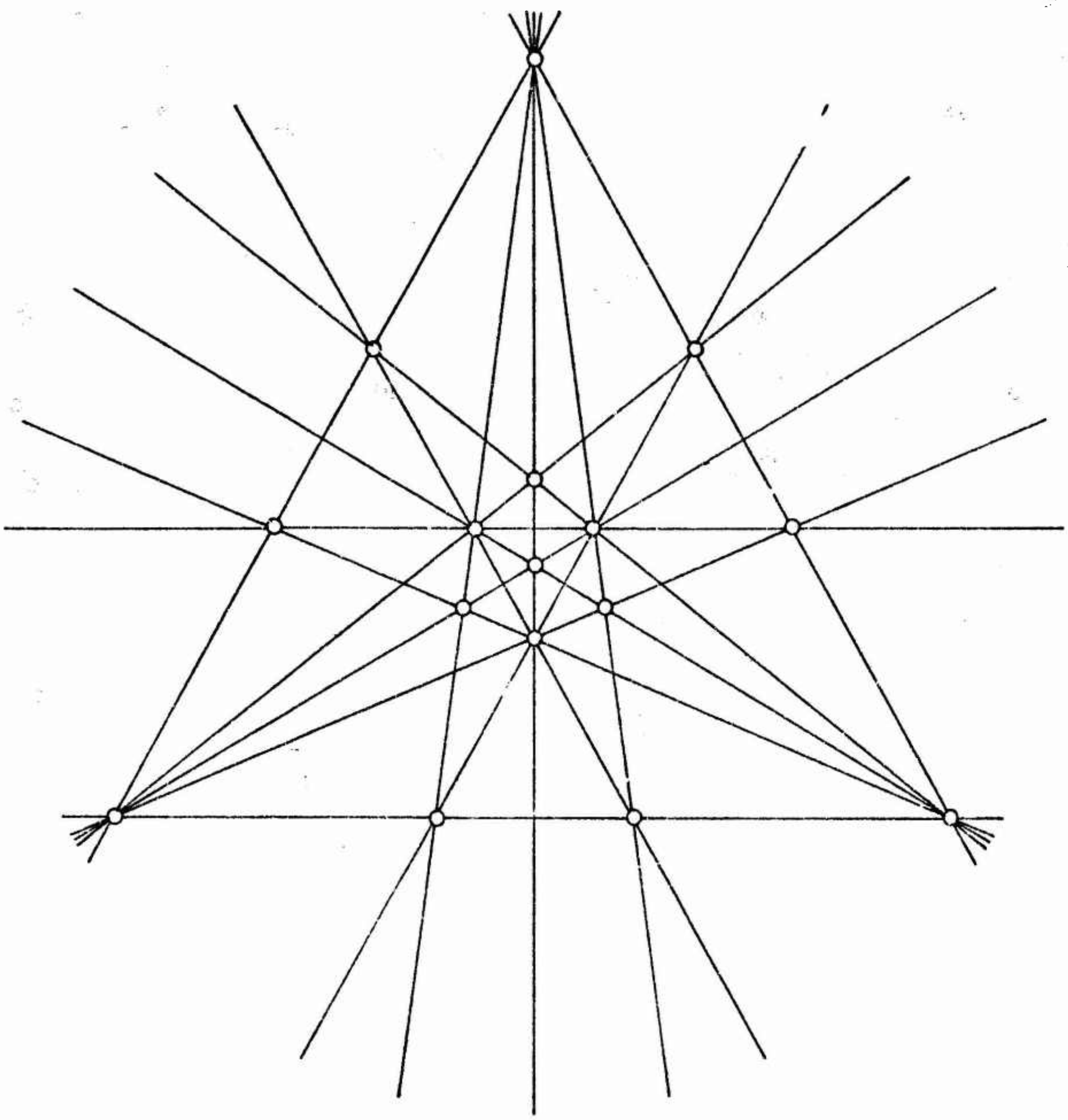

Figure 1 (a).

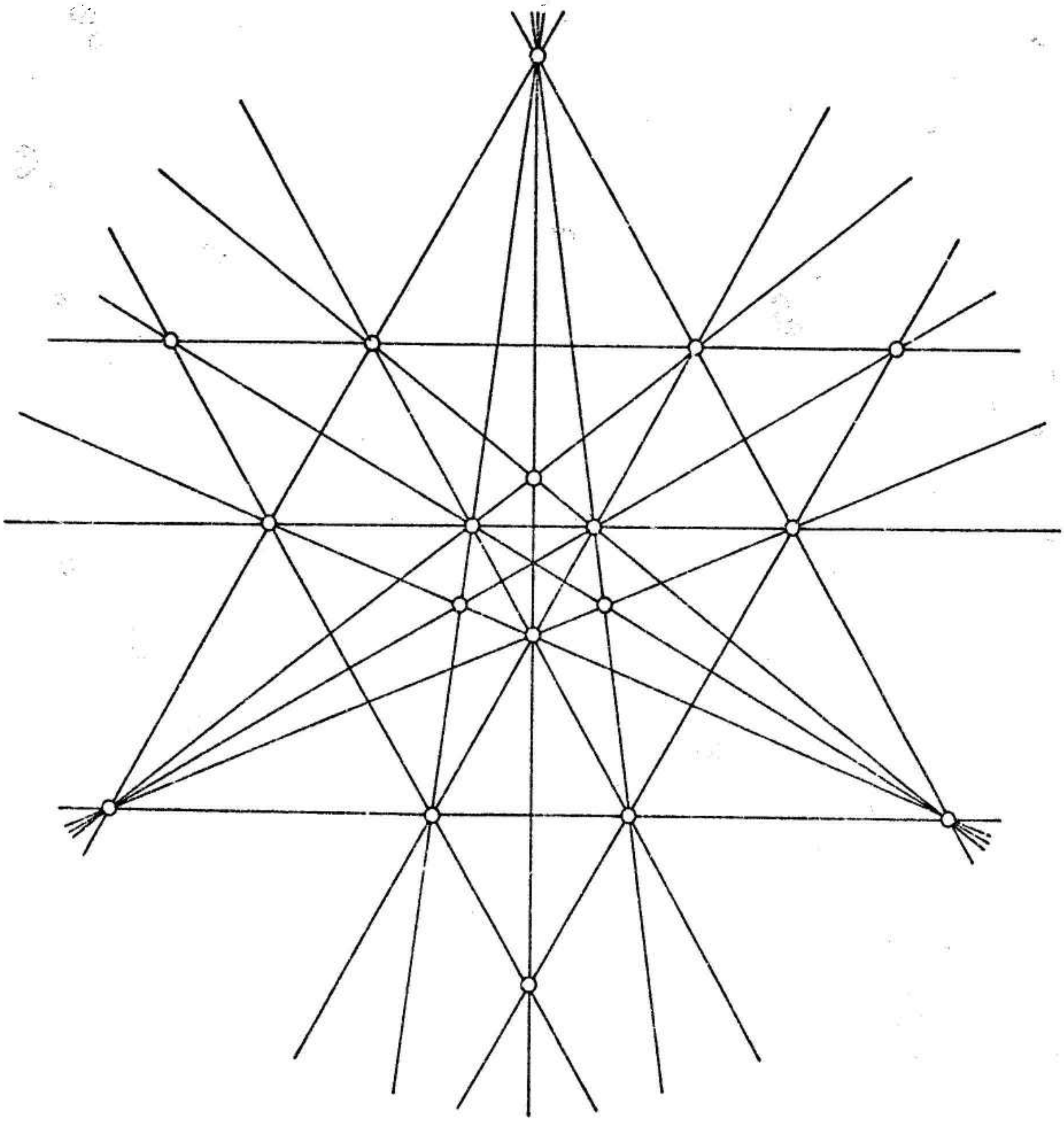

Figure 1(b).

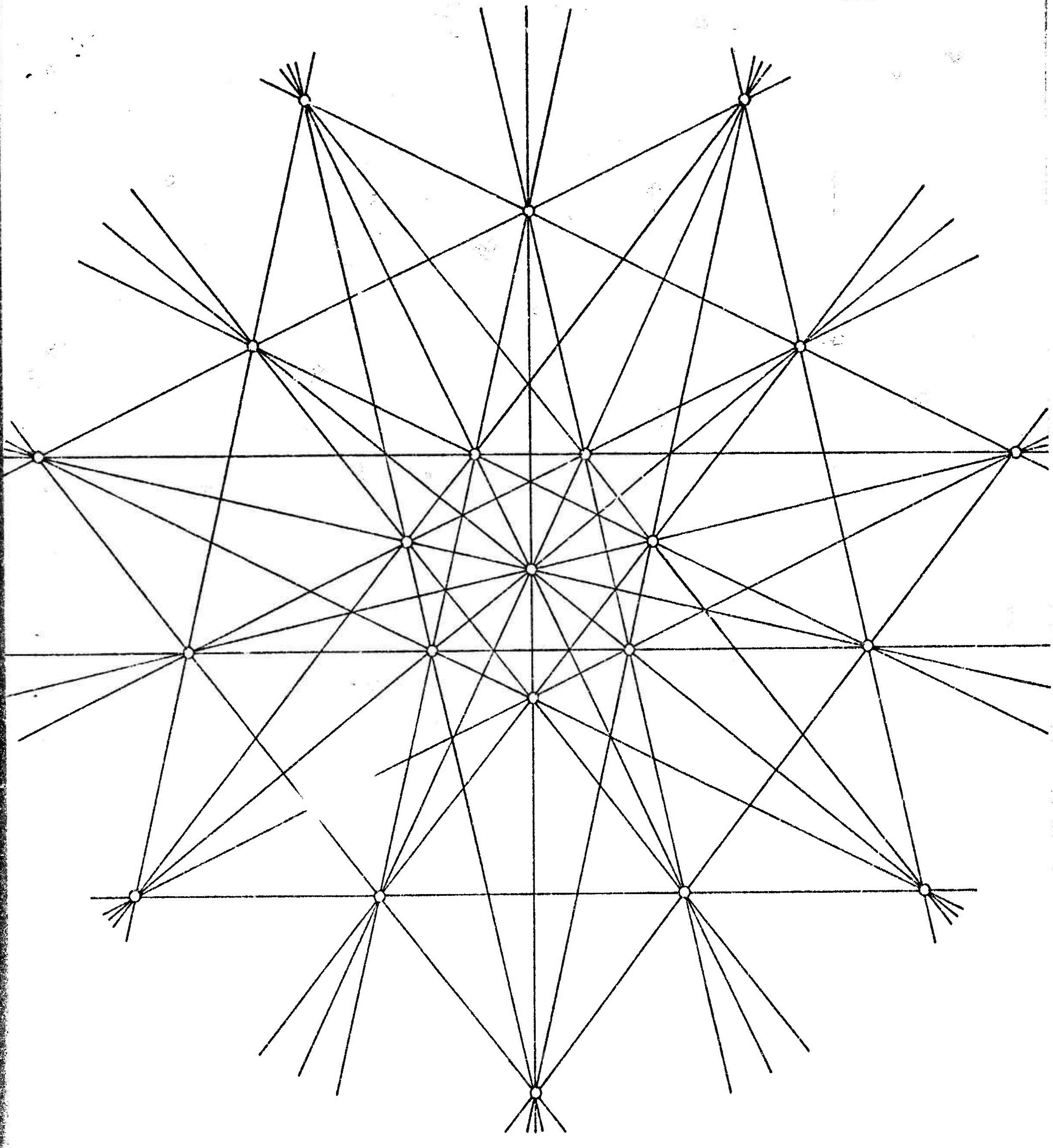

Figure 2.

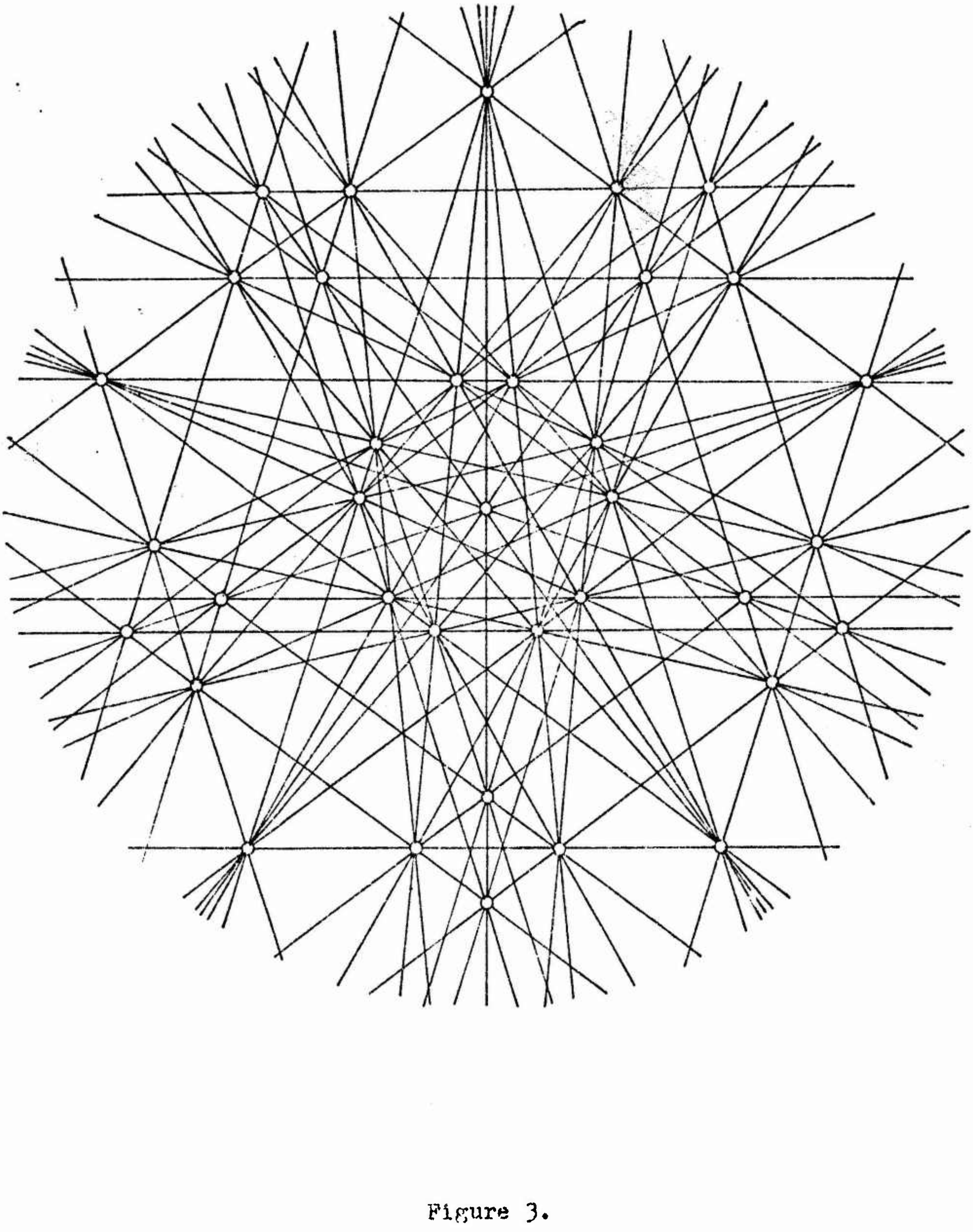

Figure 3.

actually show that $e_3(n) \geq \beta(n) \geq d_3 n^2$. If we have an arrangement $A(k,j)$ of k lines that determine $d = d_j k^{(j-1)/(j-2)}$ j-tuple points (vertices) $P_1,\ldots,P_d$, but no point is on more than j lines, we show $e_{j+1}(n) \geq d_{j+1} n^{j/(j-1)}$ as follows: In Euclidean 3-space we take one copy of $A(k,j)$ in the plane $z = 1$, and another copy of $A(k,j)$ in the plane $z = q$, for a suitable integer q, the second copy rotated by an angle α with respect to the first. Let $L_1,\ldots,L_d$ be the straight lines connecting homologous vertices in the two copies. For each $b = 2,3,\ldots,q-1$, the d intersection-points of the plane $z = b$ with the lines L_i form a set which is linearly equivalent to the set $P_1,\ldots,P_d$, and therefore lies on k suitable lines. Thus we have a family of $n = qk + d_j k^{(j-1)(j-2)}$ lines that determine $q\, d_j\, k^{(j-1)/(j-2)}$ vertices of multiplicity $j+1$ and, for a suitable choice of α, no vertex of greater multiplicity. A suitable (parallel) projection of those lines yields a planar arrangement of lines with the same property. But if q is chosen to be about $k^{1/(j-2)}$, it follows that n is about $(d_j+1)k^{(j-1)/(j-2)}$; therefore

$$e_{j+1}(n) \geq d_j\, q\, k^{(j-1)/(j-2)} \geq d_{j+1} n^{j/(j-1)},$$

and the proof is completed.

It should be noted that all the examples establishing the values of $m_4(n)$ in Table 1 actually contain no collinear quintuplets, so that the values given are at the same time lower bounds for $e_4(n)$.

A very interesting open problem is the determination of the true order of magnitude of $e_j(n)$. It is well possible that our result may be improved to $e_j(n) \geq d_j^* n^2$.

Other related open problems are:

The determination of $\tilde{m}_j(n)$, defined as the maximal number of vertices of multiplicity j possible in arrangements of n pseudolines. A few instances in which better results are available for $\tilde{m}_4(n)$ than for $m_4(n)$ are noted in Table 1.

The problem of the combinatorial analogues of $m_j(n)$ goes back to Kirkman [1847] but, as noted above, is still not completely solved even for $j = 4$.

Also open is the question of determining the maximal possible number of vertices of multiplicity j in various kinds of arrangements of curves, for example such in which each curve is simple and every two intersect in two points. The corresponding combinatorial packing problems are open as well.

* * *

The following problem was posed by de Rocquigny [1897]:

What is the maximal number $\omega_o(n)$ of points of tangency possible in a system of n mutually non-crossing circles in the plane ?

Using a blend of geometric and combinatorial arguments, Erdős-Grünbaum [1973] established that $\omega_o(n) = 3n-6$ for each $n \geq 4$, and that the same estimate holds for more general arrangements of non-crossing simple closed curves. In the same paper, partial results were obtained on the following related problems:

Determine $\omega(n)$ [and $\omega^*(n)$] , the maximal number of points of tangency in a family of n simple closed curves [circles] each two of which are either disjoint, or have one common point, or two points at which they cross each other. The results of Erdős-Grünbaum [1973] are:

<u>There exist positive constants</u> c , c_1 , c_2 <u>and</u> n_o <u>such that for all</u> $n \geq n_o$

$$c_1 n^{4/3} \leq \omega(n) \leq c_2 n^{5/3}$$

<u>and</u>

$$\omega^*(n) \geq n^{1 + c/\log\log n} .$$

The analogue of de Rocquigny's problem for spheres in 3-space is still unsolved, and so are similar problems concerning ovals in various incidence structures.

* * *

Another special class of arrangements of simple closed curves are the "independent families" and the "Venn diagrams". Motivated by the obvious and well known considerations from set thaory and logic, we shall say that a family of n simple closed curves $A_1, \ldots, A_n$ in the Euclidean plane is an <u>independent family</u> provided

$$(\#\#) \qquad X_1 \cap X_2 \cap \ldots \cap X_n \neq \emptyset$$

whenever each X_j is chosen to be either the interior or the exterior of the curve A_j. An independent family is a <u>Venn diagram</u> if and only if each of the 2^n sets in (##) is a connected region (<u>cell</u> of the Venn diagram). Examples of Venn diagrams with $n = 3$ or $n = 4$ (such as those in Figures 4a and 4b) are frequently found in the literature. Combinatorially distinct Venn diagrams of 4 congruent ellipses are possible (Figures 4c and 4d), but no complete classification is known.

Venn [1880] and many later authors (see, e.g., Gardner [1967]) asserted that there is no Venn diagram composed of ellipses for $n = 5$. This is erroneous, as is easily seen on hand of Figure 5. It is not hard to see that similar diagrams may be constructed using 5 copies of any non-circular ellipse. However, no Venn diagram can be formed by 6 ellipses. This follows at once from the case $j = 4$ of the following lemma, which is easy to establish but rather useful (see Grünbaum [1973c]).

<u>If an independent family of n curves is such that each two curves meet in at most n points, then</u>

$$j \geq (2^n - 2)/\binom{n}{2} = 4(2^{n-1} - 1)/n(n-1) .$$

Let <u>k-gon</u> mean any convex polygon with at most k sides. We shall denote by $n(k)$ the maximal number of members in any independent family of k-gons in the plane, and by $k(n)$ the minimal k such that there exists an independent family on n k-gons. The similarly defined

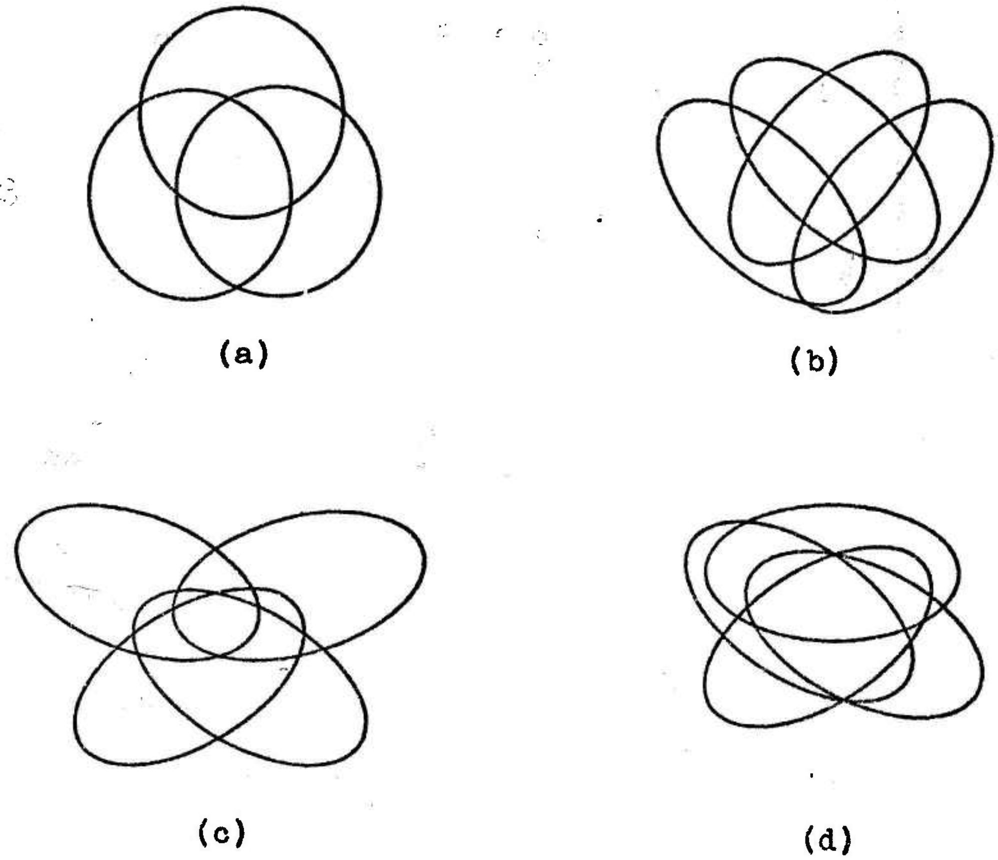

Figure 4.

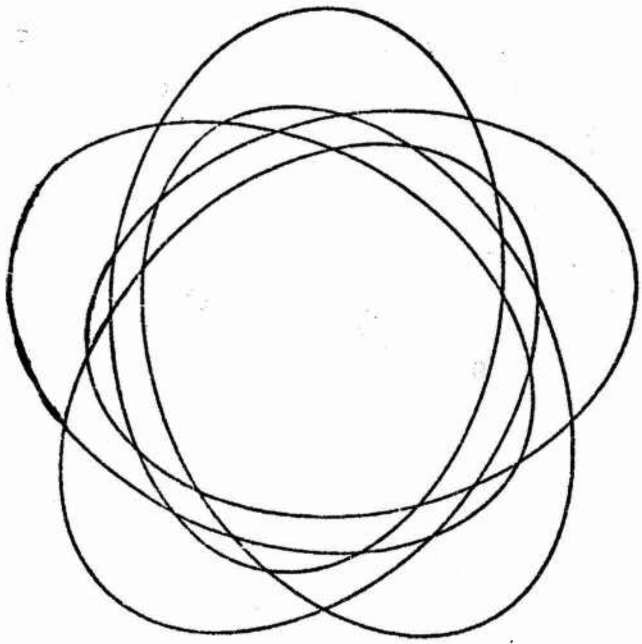

Figure 5.

numbers dealing with Venn diagrams shall be denoted $n^*(k)$ and $k^*(n)$. Then the above lemma with $j = 2k$, and an inductive construction illustrated (up to $n = 6$) in Figure 6, may be used to establish

$$\lim_{k\to\infty} n(k)/\log_2 k = \lim_{k\to\infty} n^*(k)/\log_2 k = 1 .$$

(The same result holds even if each bounded cell of the Venn diagram is required to be convex.)

The above result appears in Rényi-Rényi-Surányi [1951]; however, their proof is based on a statement obtained from our lemma by replacing the inequality of its conclusion by the stronger inequality

$$j \geq 2^{n-1}/(n-1) .$$

Unfortunately, this statement is not true, as may be seen by the example in Figure 7, where $n = j = 6$. It would be of some interest to investigate whether the stronger inequality can be established for convex polygons of $j/2$ sides , although this does not seem likely.

Among other results obtained in Grünbaum [1973c] we mention:

$$k(3) = k^*(3) = k(4) = k^*(4) = k(5) = k^*(5) = 3 ;$$

$$k(6) \leq k^*(6) \leq 4 ;$$

$$k(7) \leq 6 .$$

Equality probably holds in all those estimates; an example of a Venn diagram with 5 triangles is given in Figure 8.

Many other attractive problems on Venn diagrams, on diagrams exhibiting rotational symmetry (as in Figure 8), and on higher-dimensional generalizations are given in Grünbaum [1973c], where detailed references to the literature may also be found.

* * *

Our last topic concerns a very new problem on arrangements of simple curves in the plane, which shows that there are non-trivial problems even if only arrangements consisting of a single curve are considered. It was inspired by a recent problem of Malkevitch [1971].

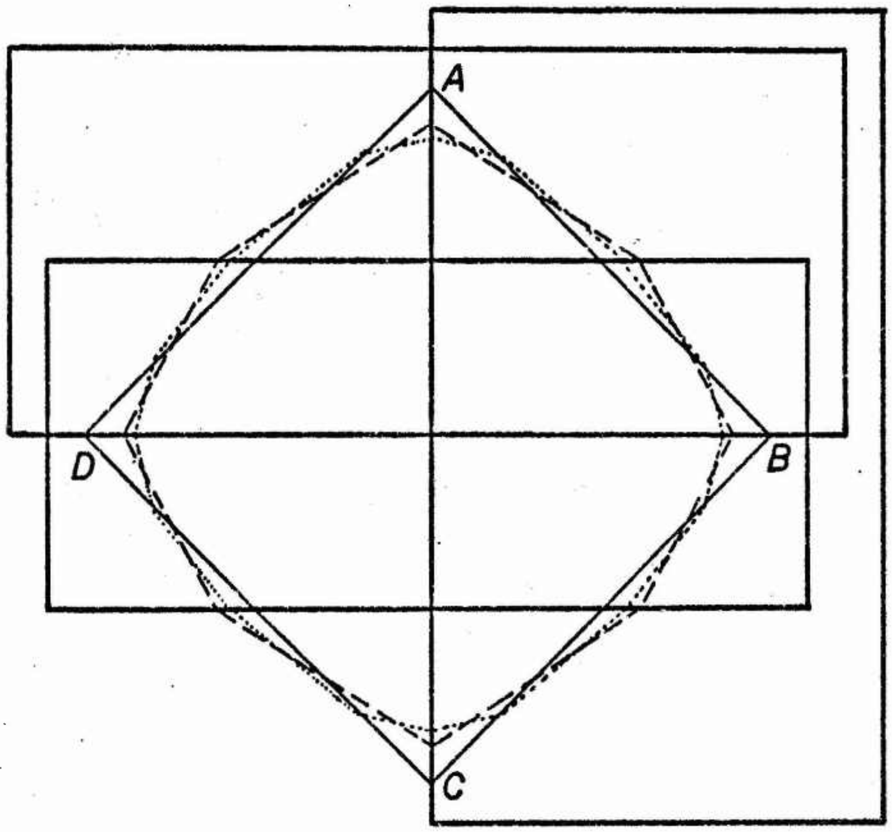

Figure 6.

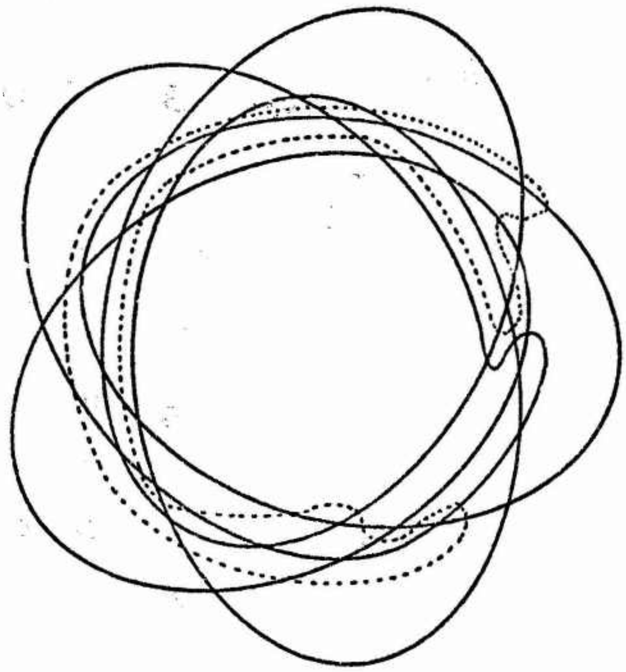

Figure 7.

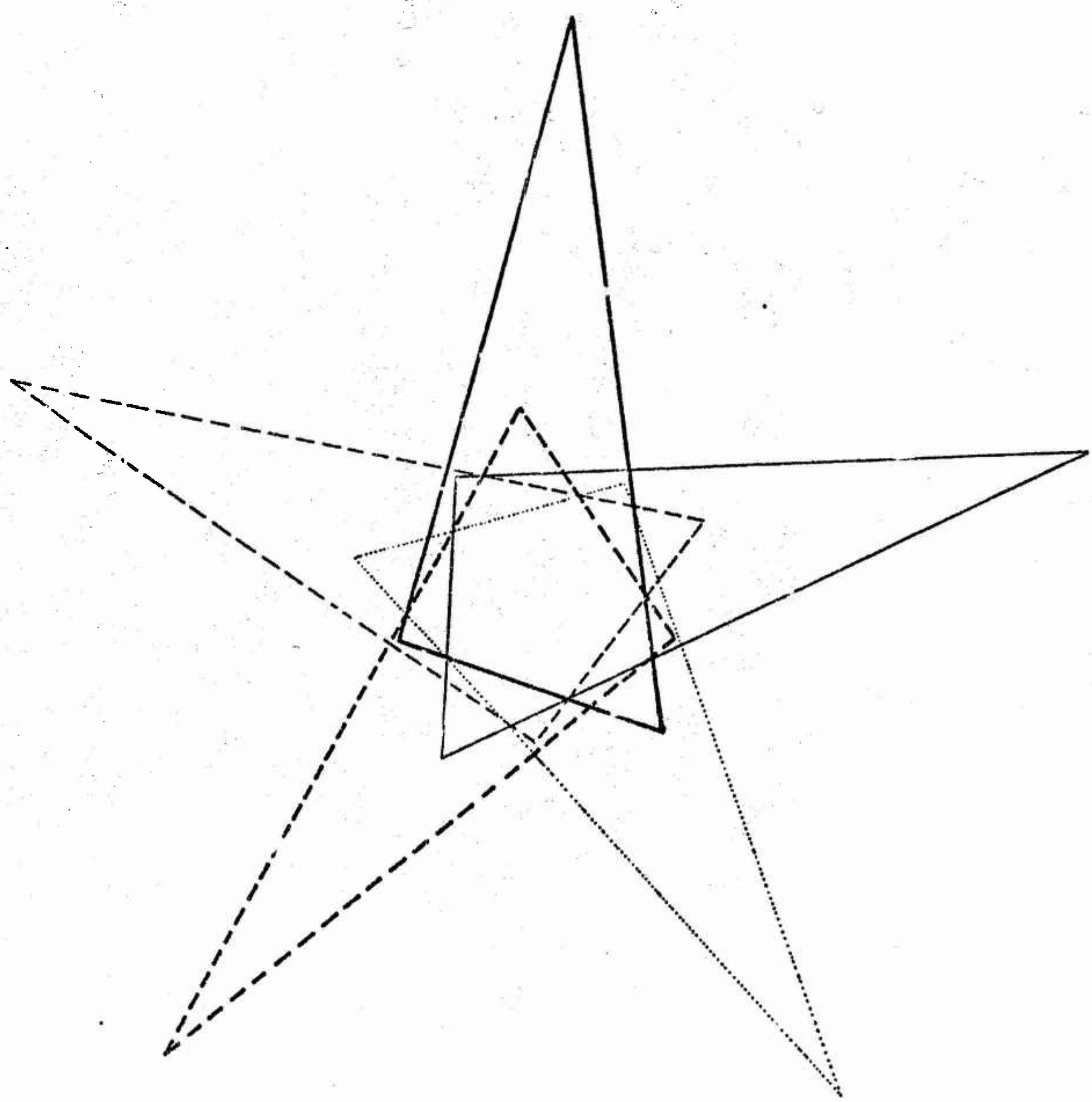

Figure 8.

Let C be a simple closed curve in the plane. We shall say that a point p is of order k with respect to C if each ray issuing from p meets C in precisely k connected components. For $k \geq 2$ we shall denote by $\mathscr{C}_k(C)$ the set of all points of order k with respect to C. The problem is to characterize, for each k, the possible sets $\mathscr{C}_k(C)$.

It is not too hard to verify the possibilities for $\mathscr{C}_k(C)$ listed in Table 2. We conjecture that Table 2 gives a complete characterization of all the possible sets $\mathscr{C}_k(C)$. This conjecture has been recently established by B. Hedman [1973] in cases $k = 2$ and $k = 3$ provided the curves C are restricted to be simple polygons. (I am indebted to B. Hedman for several corrections of my original conjecture.)

It would be very interesting to verify the conjecture for general curves C, and for $k \geq 4$. Hedman's proof relies heavily on the polygonal nature of the curves, and does not seem likely to be extendable to general C, or to $k \geq 4$.

Type of set $\mathcal{C}_k(C)$	Possible for
$\emptyset$	all $k \geq 2$
single point	all $k \geq 2$
two points	all $k \geq 2$
three points	all $k \geq 3$
four points [(a)]	$k = 3$
one segment [(b)]	all $k \geq 2$
two segments [(b),(c)]	$k = 2$
a segment and a point [(b),(d)]	all $k \geq 2$
a segment and two points [(b),(e)]	all $k \geq 3$

Table 2.

(a) For $k = 3$ possible only if the four points are the vertices of a convex quadrangle which has no pairs of parallel sides. For $k \geq 4$ possible also if the sides of one pair are parallel, or if three of the points are collinear.

(b) The segment(s) may be closed, open, or half-open.

(c) Possible only if the segments are disjoint, non-parallel, and the convex hull of their union is a quadrangle.

(d) Possible only if the point is not collinear with the segment.

(e) Possible only if the two points are in the same open halfplane determined by the segment.

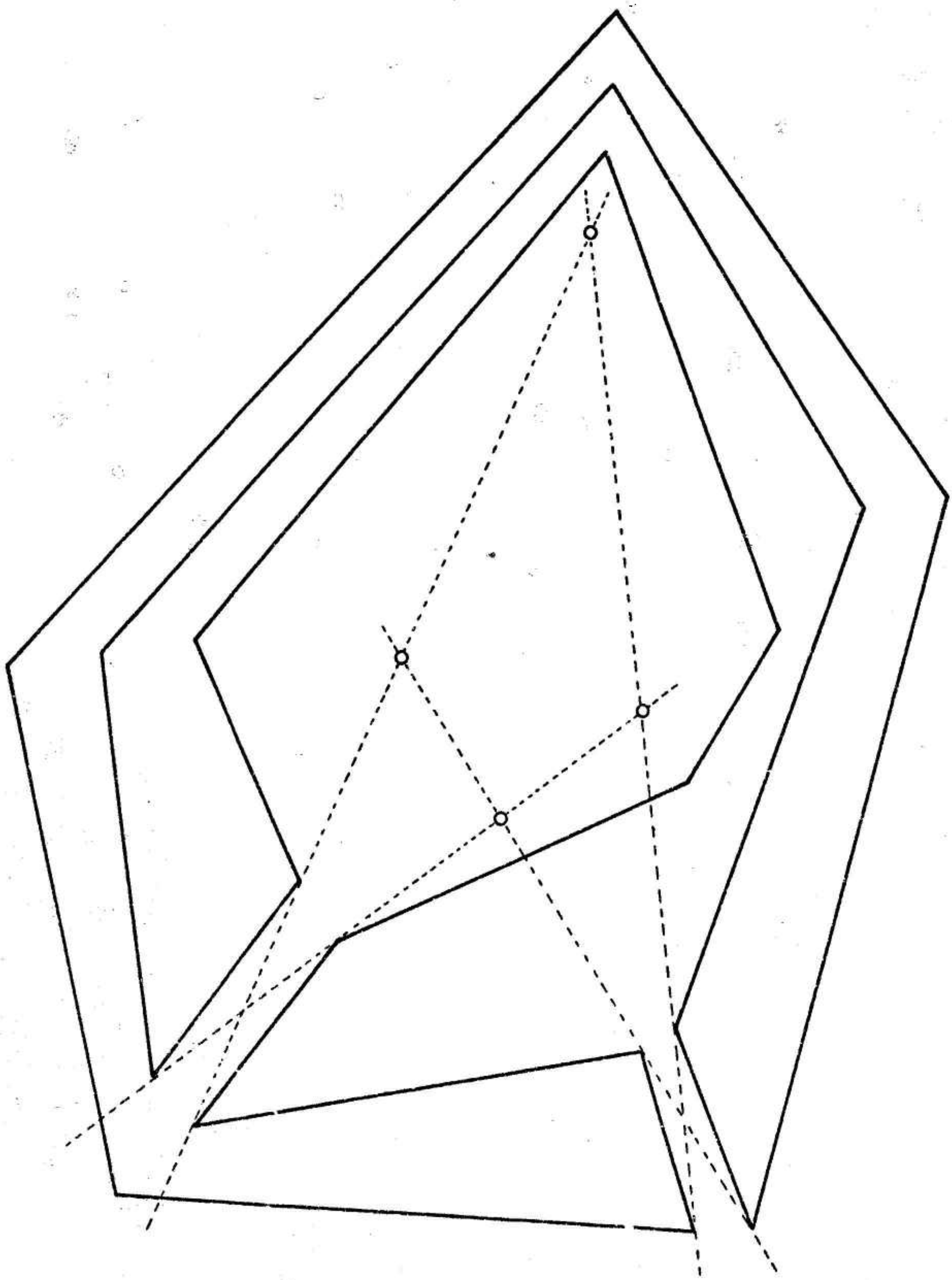

Figure 9.

A polygonal curve C for which $\mathscr{C}_3(C)$ consists of 4 points.

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